Handout based on "Why is maximum clique often easy in practice?" by Jose L. Walteros and Austin Buchanan which is to appear at *Operations Research*.

Definition 1. *A graph G is d-degenerate if every induced subgraph $G[S]$ with $S \neq \emptyset$ has a vertex v with $\deg_{G[S]}(v) \leq d$. The degeneracy of G is the smallest value of d for which G is d-degenerate.*

$\delta(G[S]) \leq d$

Lemma 1 (Lick and White, 1970). *A graph is d-degenerate if and only if its vertices can be ordered $(v_1, v_2, \ldots, v_n)$ so that every vertex v_i in the ordering satisfies* $\underbrace{|N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\}|}_{=: \, rdeg(v_i)} \leq d$.

($\Rightarrow$)

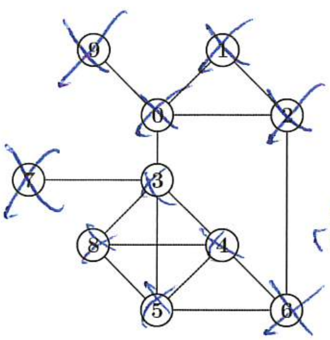
$G' \leftarrow G$
For $i = 1, 2, \ldots, n$
  pick a vertex $\underbrace{v \text{ of min degree in } G'}_{d \geq rdeg(v) = |N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\}|}$
  $v_i \leftarrow v$
  $G' \leftarrow G - v_i$
return $(v_1, v_2, \ldots, v_n)$

$O(n+m)$

($\Leftarrow$) Suppose $(v_1, v_2, \ldots, v_n)$ satisfies $|N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\}| \leq d$ for each v_i, and let $G[S]$ be an induced subgraph. Let $v = v_i$ be the earliest vertex from S in the ordering. Then,

$$\deg_{G[S]}(v) \leq |N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\}| \leq d. \quad \square$$

Compute a Minimum Degree (MD) ordering and the right-degree of each vertex in it.



	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
$v=$	7	9	1	2	0	6	3	4	5	8
$rdeg(v)=$	1	1	2	2	1	2	3	2	1	0

1-core

3-core

2-core

What is the degeneracy d, the clique number ω, and the clique-core gap $g := (d+1) - \omega$?

$\omega \leq \chi \leq d+1$

$\omega = 4$

$d = 3$

$g = 0$

tests if $w(G) \geq (d+1)-p$?

i.e., is $g \leq p$?

main(G, p)

$n+m$

1. compute an MD ordering $(v_1, v_2, \ldots, v_n)$ and degeneracy d of G;
2. let $D = \{v_i \in V \mid i \leq n-d,\ \mathrm{rdeg}(v_i) \geq d-p\}$; ← candidate subproblems
3. for $v_i \in D$ do

   $|V_i| \leq d$

   d^2 → (a) construct $\overline{G}[V_i]$, where V_i is the right-neighborhood of v_i;

   (b) if $\overline{G}[V_i]$ has a vertex cover of size $q_i := |V_i| + p - d$, return "yes";

   $q_i \leq p$

4. construct $\overline{G}[V_f]$, where $V_f = \{v_f, \ldots, v_n\}$ and $f := n - d + 1$;
5. if $\overline{G}[V_f]$ has a vertex cover of size $q_f := p - 1$, return "yes";
6. return "no."

k-vertex cover in an n-vertex graph, takes $T(n,k) = 1.2738^k + kn$ Chen et al. (2010)

Theorem 1. *When p is a constant, algorithm* **main** *runs in time $O(dm) = O(m^{1.5})$.*

Let $G' = (V', E')$ be the $(d-p)$-core of G, i.e., the largest subgraph of G in which $\delta(G') \geq d-p$.

Let $n' = |V'|$ and $m' = |E'|$.

Then

$$2m' = \sum_{v \in V'} \deg_{G'}(v) \geq n'(d-p). \qquad (*)$$

So,

$$d|D| \leq dn' \underset{\text{by } *}{\leq} 2m' + pn'. \qquad (**)$$

Thus the total time is

$$O\Big(n+m + (|D|+1)d^2 + \sum_{v_i \in D} T(|V_i|, q_i) + T(|V_f|, q_f)\Big)$$

$$= O\Big(n+m + \Big(\frac{2m'+pn'+d}{\cancel{d}}\Big)d^{\cancel{2}} + (|D|+1)(1.2738^p + pd)\Big)$$

$$= O\big(n+m + (2m'+pn'+d)d + n'd\big)$$

$$= O(m + m'd + n'd) = O(dm) = O(m^{1.5}). \quad \square$$